Chapter 5
Homogeneous Representations of Points, Lines and Planes

This chapter motivates and introduces homogeneous coordinates for representing geometric entities. Their name is derived from the homogeneity of the equations they induce. Homogeneous coordinates represent geometric elements in a projective space, as inhomogeneous coordinates represent geometric entities in Euclidean space. Throughout this book, we will use Cartesian coordinates: inhomogeneous in Euclidean spaces and homogeneous in projective spaces. A short course in the plane demonstrates the usefulness of homogeneous coordinates for constructions, transformations, estimation, and variance propagation. A characteristic feature of projective geometry is the symmetry of relationships between points and lines, called duality. In this chapter we aim at exploiting the algebraic properties of the representations of geometric entities and at giving geometrically intuitive interpretations.

5.1 Homogeneous Vectors and Matrices

5.1.1 Definition and Notation

Definition 5.1.2: *Homogeneous coordinates (J. Plücker 1829).* Homogeneous coordinates $\mathbf{x}$ of a geometric entity χ are invariant with respect to multiplication by a scalar $\lambda \neq 0$: thus $\mathbf{x}$ and $\lambda\mathbf{x}$ represent the same entity χ. $\diamond$

We will find homogeneous representations for geometric entities, such as points, lines and planes, but also for transformations. The homogeneous representation is not unique, as $\lambda \neq 0$ can be chosen arbitrarily; $\mathbf{x}$ and $-\mathbf{x}$ represent the same entity. Uniqueness of the entity is guaranteed as long as not all coordinates vanish, thus $|\mathbf{x}| \neq 0$.

In certain applications it will be useful to restrict the freedom of scaling and to distinguish between opposite directions, e.g., when reasoning about the left or the right side of an entity or when modelling a real camera: points always have to be located in front of a camera, and this needs to be reflected in the modelling. This leads to *oriented* entities whose representation is only invariant to the multiplication with a positive scalar.

It will occasionally be useful to reduce the ambiguity of the scaling and normalize homogeneous entities. We will distinguish between spherical normalization, which we denote by the index s, e.g. $\mathbf{x}^s$ where $|\mathbf{x}^s| = 1$, and similarly the Euclidean normalization $\mathbf{x}^e$ where some of the elements of $\mathbf{x}^e$ can be interpreted as elements in Euclidean space.

Due to these representational properties, we need to clarify the usage of the equal sign "=" in the context of homogeneous entities. It has three uses:

1. The equal sign is used to indicate equality, following the convention in mathematics.
2. The equal sign is used to indicate a value assignment as in some computer languages. For example, $\mathbf{l} = \mathbf{x} \times \mathbf{y}$ is read as the vector $\mathbf{l}$ is to be determined as the cross product of $\mathbf{x}$ and $\mathbf{y}$. This is sometimes written as $\mathbf{l} := \mathbf{x} \times \mathbf{y}$.
3. The equal sign is used to state that the representations on the left and the right-hand sides refer to the same object. Thus the two representations are equal up to scaling. The equation above $\mathbf{l} = \mathbf{x} \times \mathbf{y}$ (a homogeneous relation), thus can be read as a condition for the line parameters $\mathbf{l}$ to be equal to the parameters of the line connecting the points $\mathcal{x}(\mathbf{x})$ and $\mathcal{y}(\mathbf{y})$. This sometimes is written as $\mathbf{l} \cong \mathbf{x} \times \mathbf{y}$, or as $\lambda \mathbf{l} = \mathbf{x} \times \mathbf{y}$ with some $\lambda \neq 0$, making the scale factor explicit.

We will use the simple equal sign and only specify the relations 2 and 3 if the context requires.

In contrast to inhomogeneous entities such as l, X, and R, homogeneous entities are designated with upright letters, such as $\mathbf{l}$, $\mathbf{X}$, and P. Planes are designated with letters from the beginning of the alphabet, lines with letters from the middle of the alphabet and points with letters from the end of the alphabet. Points and lines in the plane will be called 2D points and 2D lines, in contrast to 3D points and 3D lines in space.

We distinguish between *names* and *representations* of geometric entities. The symbol $\mathcal{X}$ denotes the name of the point whereas its coordinates are denoted by X or $\mathbf{X}$; thus, we can write $\mathcal{X}(X)$ or $\mathcal{X}(\mathbf{X})$ depending on our assumptions about the representation of the point $\mathcal{X}$. The notations used are collected in Tables 5.1 and 5.2.

Table 5.1 Names of basic geometric entities in 2D and 3D

element	2D	3D
planes		$\mathcal{A}$, $\mathcal{B}$, ...
lines	l, m, ...	$\mathcal{L}$, $\mathcal{M}$, ...
points	$\mathcal{x}$, $\mathcal{y}$, ...	$\mathcal{X}$, $\mathcal{Y}$, ...

Table 5.2 Notation for inhomogeneous and homogeneous vectors and matrices

	2D	3D	transformations
inhomogeneous	x	X	R
homogeneous	$\mathbf{l}$, $\mathbf{x}$	$\mathbf{A}$, $\mathbf{L}$, $\mathbf{X}$	H

Homogeneous coordinates have a number of advantages which make them indispensable in our context:

- They allow us to represent entities at infinity, which occurs frequently, e.g., when dealing with vanishing points. Conceptually, homogeneous coordinates are the natural

representation of elements of a projective space, by which we mean the corresponding Euclidean space together with the elements at infinity of all lines in that plane.

- Homogeneous coordinates allow us to easily represent straight line-preserving transformations, thus not only translations, rotations or affine transformations but also projective transformations, e.g., when representing the mapping from 3D object space to 2D image space in a pinhole camera.
- They simplify concatenation and inversion of straight line-preserving transformations, since all transformations are represented as a matrix vector product.
- They simplify the construction of geometric elements from given ones as well as the expression of geometric constraints as sets of homogeneous equations.
- All geometric operations, constructions, and transformations are bilinear forms. As a consequence, the uncertainty of vectors and matrices using covariance matrices can easily be propagated, as the necessary Jacobians are derived without effort.

We will first introduce the basic ideas in a short course, motivating the content of the chapter, and then discuss the individual concepts in detail.

5.1.2 A Short Course

This subsection is meant to give an intuitive introduction to the use of homogeneous coordinates in 2D space and to exemplify their advantages for points and lines and their relations and transformations.

5.1.2.1 Representation with Homogeneous Coordinates

The Hessian normal form of a straight line l in the xy-plane is given by *Hessian normal form*

$$x \cos \phi + y \sin \phi - d = 0 \,, \tag{5.1}$$

see Fig. 5.1. Whereas the point χ is represented with its inhomogeneous coordinates $\boldsymbol{x} = [x, y]^\mathsf{T}$, thus $\chi(x, y)$, the line l is represented with the Hessian coordinates $\boldsymbol{h} = [\phi, d]^\mathsf{T}$, namely the direction ϕ of its normal in mathematically positive, i.e., counterclockwise, direction counted from the x-axis, and its distance d from the origin O,[1] thus $l(\phi, d)$.
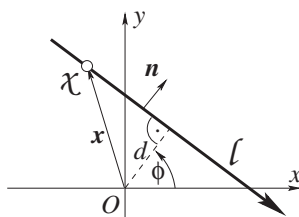
Fig. 5.1 Straight line with parameters of Hessian normal form. The normal direction $\boldsymbol{n}$ of the line points to the left w.r.t. the direction (or orientation, cf. Sect. 9.1.1.3, p. 346) of the line

Equation (5.1) may be written in different forms and allows different interpretations:

- The equation represents the incidence of the point $\chi(\boldsymbol{x})$ with the line $l(\boldsymbol{h})$. This symmetric incidence relation $\iota(\chi, l)$ is equivalent to the dual relations: "The point χ lies on the line l" and "The line l passes through the point χ".
- The equation may be written as

---

[1] We assume the distance is measured in the direction of the normal.

$$\boldsymbol{n}^{\mathsf{T}}\boldsymbol{x} = d \qquad \text{with} \qquad \boldsymbol{n} = \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} \tag{5.2}$$

if we use the normal vector $\boldsymbol{n}$. It suggests the line is to be represented by three parameters, $[n_x, n_y, d]^{\mathsf{T}}$. However, they satisfy one constraint, namely $|\boldsymbol{n}| = 1$. The representation with $\boldsymbol{n}$ does not have a singularity when estimating the direction $\boldsymbol{n}$, unlike the angle representation of the direction with ϕ (see the discussion on using quaternions for representing rotations, Sect. 8.1.5.2, p. 335). This has significant advantages.

- The equation may be written as

$$\mathbf{x}^{e\mathsf{T}}\mathbf{l}^e = 0 \qquad \text{with} \qquad \mathbf{x}^e = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad \mathbf{l}^e = \begin{bmatrix} \cos\phi \\ \sin\phi \\ -d \end{bmatrix}. \tag{5.3}$$

This suggests that both the point and the line are to be represented with 3-vectors, thus $\chi(\mathbf{x}^e)$ and $\ell(\mathbf{l}^e)$. They are homogeneous vectors, as multiplying them with an arbitrary scalar $\neq 0$ does not change the incidence relation. But they are normalized in a well-defined way, namely such that the inhomogeneous parameters (x, y) and d can be directly inferred. We will discuss normalization below.

Moreover, the equation is symmetric in $\mathbf{x}$ and $\mathbf{l}$ as $\mathbf{x}^{\mathsf{T}}\mathbf{l} = \mathbf{l}^{\mathsf{T}}\mathbf{x} = 0$, which algebraically reflects the symmetry of the incidence property $\iota(\chi, \ell)$.

- The equation may more generally be written as

$$\mathbf{x}^{\mathsf{T}}\mathbf{l} = 0 \tag{5.4}$$

with the vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = w \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \tag{5.5}$$

$$\mathbf{l} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \pm\sqrt{a^2 + b^2} \begin{bmatrix} \cos\phi \\ \sin\phi \\ -d \end{bmatrix}. \tag{5.6}$$

The factors $w \neq 0$ and $|[l_1, l_2]| = \sqrt{a^2 + b^2} \neq 0$ can be chosen arbitrarily. Therefore, points and lines can be represented by nearly arbitrary 3-vectors, namely by restricting the absolute value of $w = x_3$ and the absolute value of $[a, b]^{\mathsf{T}} = [l_1, l_2]^{\mathsf{T}}$ not to be zero. As the relation (5.4) is a homogeneous equation, the corresponding representations of the points are homogeneous, and the 3-vectors $\mathbf{x}$ and $\mathbf{l}$ are called the homogeneous coordinates of the point χ and the line ℓ respectively.

homogeneous coordinates

We can easily determine the Euclidean representation of the point and the line from

$$x = \frac{u}{w} \quad y = \frac{v}{w}, \qquad\qquad \phi = \operatorname{atan2}(b, a) \quad d = -\frac{c}{\sqrt{a^2 + b^2}} \tag{5.7}$$

or

$$\boldsymbol{x} = \frac{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}{x_3}, \qquad\qquad \phi = \operatorname{atan2}(l_2, l_1) \quad d = -\frac{l_3}{\left\|\begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\right\|}. \tag{5.8}$$

5.1.2.2 Normalizations

Homogeneous coordinates of a point or a line are not unique. Uniqueness may be achieved by normalization, i.e., by fixing the scale factor. Two types of normalizations are common, Euclidean and spherical.

Euclidean Normalization. By *Euclidean normalization* the vector is transformed such that the Euclidean properties become visible (Fig. 5.2). We obtain

$$\mathbf{x}^e = \mathsf{N}^e(\mathbf{x}) = \frac{1}{w} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix}, \qquad \mathbf{l}^e = \mathsf{N}^e(\mathbf{l}) = \frac{1}{\sqrt{a^2+b^2}} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \boldsymbol{n} \\ -d \end{bmatrix}. \qquad (5.9)$$

Therefore, following Brand (1966), we introduce the following notation for points and lines to specify the Euclidean part and the homogeneous part of a homogeneous vector. The Euclidean part, indexed by 0, implicitly contains the Euclidean properties: for points the two coordinates, and for lines the distance to the origin,

$$\mathbf{x} = \begin{bmatrix} \boldsymbol{x}_0 \\ x_h \end{bmatrix}, \qquad \mathbf{l} = \begin{bmatrix} \boldsymbol{l}_h \\ l_0 \end{bmatrix}. \qquad (5.10)$$

Euclidean normalization then reads as

$$\mathbf{x}^e = \frac{\mathbf{x}}{x_h}, \qquad \mathbf{l}^e = \frac{\mathbf{l}}{|\boldsymbol{l}_h|}. \qquad (5.11)$$

Spherical Normalization. By *spherical normalization* all coordinates of a homogeneous vector are processed the same way and the complete vector is normalized to 1 (Fig. 5.3). The spherically normalized homogeneous coordinates of a 2D point χ and of a 2D line ℓ are

$$\mathbf{x}^s = \mathsf{N}(\mathbf{x}) = \frac{1}{\sqrt{u^2+v^2+w^2}} \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \qquad \mathbf{l}^s = \mathsf{N}(\mathbf{l}) = \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \qquad (5.12)$$

Thus the spherically normalized homogeneous coordinates of all 2D points and 2D lines build the unit sphere S^2 in $\mathbb{R}^3$.
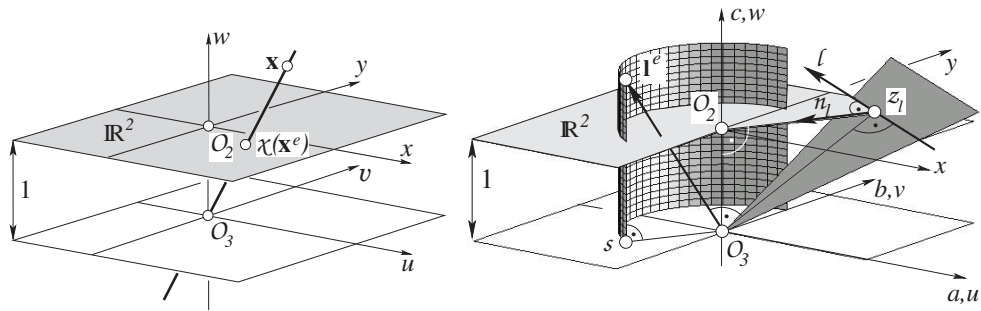
We will frequently use spherically normalized homogeneous vectors. They have several advantages:

1. They lie on a sphere, which is a closed manifold without any borders. Thus geometrically, i.e., if we do not refer to a special coordinate system, there are no special points in the projective plane.
2. The redundancy in the representation – we use three coordinates for a 3D entity – requires care in iterative estimation procedures, as the length constraint needs to be introduced. Iteratively correcting spherically normalized vectors can be realized in the tangent space which for 2D points is a tangent plane at the spherically normalized vector.
3. As the points $\mathbf{x}^s$ and the point $-\mathbf{x}^s$ represent the same 2D point, the representation is not unique. Taking these two points as two different ones leads to the concept of oriented projective geometry, which among other things can distinguish between lines with different orientation (Chap. 9, p. 343).

5.1.2.3 Geometric Interpretation of Homogeneous Coordinates and the Projective Plane

The last two paragraphs suggest an intuitive and important geometric interpretation of homogeneous coordinates as embedding the real plane $\mathbb{R}^2$ with origin O_2 and axes x and y into the 3D Euclidean space $\mathbb{R}^3$ with origin O_3 and axes u, v and w, cf. Fig. 5.2, left. The Euclidean normalized coordinate vector $\mathbf{x}^e = [u, v, w]^\mathsf{T} = [x, y, 1]^\mathsf{T}$ lies in the plane $w = 1$. *Euclideanly* The origin O_2 has coordinates $\mathbf{x}_{O_2} = [0, 0, 1]^\mathsf{T}$. The u- and the v-axes are parallel to the *normalized vector* x- and the y-axes respectively. Thus, adding the third coordinate, 1, to an inhomogeneous coordinate vector $\boldsymbol{x}$ to obtain $\mathbf{x}^e$ can be interpreted as embedding the real Euclidean plane

Fig. 5.2 Representation with homogeneous coordinates, Euclidean normalization. **Left:** 2D point. The real plane $\mathbb{R}^2$ is embedded into the 3D space $\mathbb{R}^3$ with coordinates u, v and w. Any vector $\mathbf{x}$ on the line joining the origin O_3 of the (uvw)-coordinate system and the point χ, except the origin itself, can represent the point χ on the Euclidean plane $\mathbb{R}^2$. The intersection of the line χO_3 with the plane $w = 1$ yields the Euclideanly normalized homogeneous coordinate vector $\mathbf{x}^e$ of χ. **Right:** 2D line. The real plane $\mathbb{R}^2$ is embedded into the 3D space $\mathbb{R}^3$. Coordinates a, b and c are used to represent 2D lines. The 2D line l is represented by the normal $\mathbf{l}$ of the plane passing through the origin O_3 and the line. When the Euclideanly homogeneous coordinates $\mathbf{l}^e$ (5.9), p. 199 are used, their first two elements are normalized to 1, and the vector $\mathbf{l}^e$ lies on the unit cylinder (distance $sO_3 = 1$) parallel to the w-axis. The distance of the line l from the origin O_2, which is in the direction of the normal n_l, is identical to the c-component of $\mathbf{l}^e$

$\mathbb{R}^2$ into $\mathbb{R}^3$. Points with coordinates $\mathbf{x} = \lambda \mathbf{x}^e$ are on a straight line through the origin O_3. They represent the same point, namely χ. You can also say: the straight line χO_3, taking χ, which is embedded in the 3D (uvw)-space, represents the homogeneous point χ.

A similar geometric interpretation can be given for lines. Here, we embed the real plane $\mathbb{R}^2$ into $\mathbb{R}^3$, but with an (a, b, c)-coordinate system at O_3.

The vector $\mathbf{l}^e := [a, b, c]^\mathsf{T} = [\cos\phi, \sin\phi, -d]^\mathsf{T}$ lies on the vertical cylinder $a^2 + b^2 = 1$ with unit radius, see Fig. 5.2, right. The vector $\mathbf{l}^e$ is the normal of the plane through O_3 and l, as $\mathbf{x}^\mathsf{T} \mathbf{l}^e = 0$ for all points on l. The coordinate d of this vector is equal to the distance of the line l from the origin O_2, as can be proven geometrically by investigating

Exercise 5.13

the coplanar triangles (O_2, z_l, O_3) and $(s, \mathbf{l}^e, O_3)$.

The spherically normalized homogeneous coordinates can be geometrically interpreted

spherically normalized vectors

in a similar way.

The point $\mathbf{x}^s$ lies on the unit sphere S^2 in the three-dimensional (uvw)-space $\mathbb{R}^3$, see Fig. 5.3, left. Obviously, the negative vector $-\mathbf{x}^s$, also representing the point χ, lies on the unit sphere. All points on the unit sphere S^2, except those on the equator $u^2 + v^2 = 1$, represent points of $\mathbb{R}^2$.

The points on the equator have a well-defined meaning: when a point χ moves away from the origin O_2 towards infinity, its spherically normalized homogeneous vector moves towards the equator. Thus, points on the equator of S^2 represent points χ_∞ at infinity.

point at infinity

They are represented by homogeneous coordinate vectors with $w = 0$, independently of their normalization.

If we take the union of all points in the Euclidean plane $\mathbb{R}^2$ and all points at infinity,

projective plane

we obtain what is called the *projective plane* $\mathbb{P}^2$. Both can be represented by the unit sphere, with opposite points identified.

The point $\mathbf{l}^s$ also lies on the unit sphere in the three-dimensional (abc)-space $\mathbb{R}^3$, see Fig. 5.3, right. It is the unit normal of the plane through O_3 and the line l. This plane intersects the unit sphere in a great circle.

polarity on the sphere

The relation between this circle and the normal $\mathbf{l}^s$ is called *polarity on the sphere*: the point $\mathbf{l}^s$ is what is called the *pole* of the circle; the circle is the *polar* of the point $\mathbf{l}^s$.

If a line moves towards infinity, its homogeneous vector moves towards the c-axis. Therefore, the origin O_2 or its antipode represent the line l_∞ at infinity. Since lines are dual to points, cf. below, this unit sphere S^2 represents the dual projective plane.

This visualization of the projective plane is helpful for understanding certain constructions and will be used throughout.
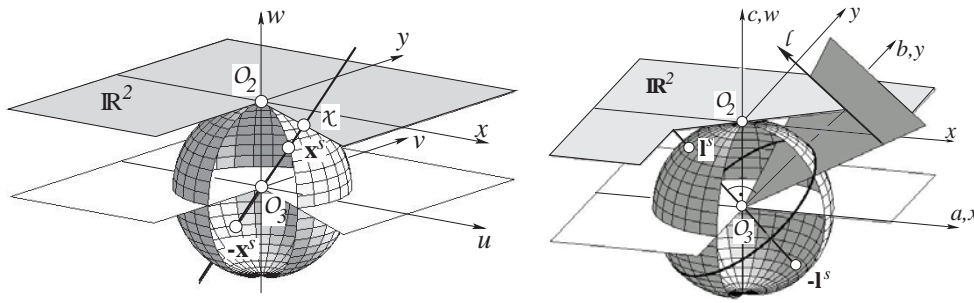
Fig. 5.3 Spherical normalization. **Left:** 2D point. The spherically normalized point $\mathbf{x}^s$ lies on the upper hemisphere of the unit sphere S^2. The antipode $-\mathbf{x}^s$ also represents the point χ. Points on the equator $u^2 + v^2 = 1$ represent points at infinity. **Right:** 2D line l lying in $\mathbb{R}^2$. The spherically normalized homogeneous vector $\mathbf{l}^s$ is the unit normal of the plane through O_3 and l. When seeing the plane $(O_3 l)$ from $\mathbf{l}^s$, the origin O_3 is on the left side of the line l. Therefore, the antipode point $-\mathbf{l}^s$ represents the same line, however, with opposite direction

5.1.2.4 Line Joining Two Points, Intersection of Two Lines, and Elements at Infinity

Now, let us determine the line

$$l = \chi \wedge y \tag{5.13}$$

joining two points χ and y. The symbol $\wedge$ (read: wedge) indicates the join. If the two points are given with their homogeneous coordinates, thus $\chi(\mathbf{x})$ and $y(\mathbf{y})$, the joining line is given by

$$l = \chi \wedge y : \quad \mathbf{l} = \mathbf{x} \times \mathbf{y} = \mathsf{S}(\mathbf{x})\mathbf{y}, \tag{5.14}$$

as then the vector $\mathbf{l}$ is perpendicular to $\mathbf{x}$ and $\mathbf{y}$, thus $\mathbf{x}^\mathsf{T}\mathbf{l} = 0$ and $\mathbf{y}^\mathsf{T}\mathbf{l} = 0$; thus, the line passes through both points. Matrix $\mathsf{S}(\mathbf{x})$ is the skew symmetric matrix induced by the 3-vector $\mathbf{x}$,

Exercise 5.1

$$\mathsf{S}(\mathbf{x}) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}. \tag{5.15}$$

A first remark on notation: The symbol for the join of two geometric entities is not unique in the literature. The wedge sign "$\wedge$" often is used for the cross product in physics. This is the reason for using it here for the join of two points, as the homogeneous coordinates of the resulting line is obtained by the cross product. Observe: some authors use the sign $\vee$ for the join of two points.

We will overload the symbol in two ways: (1) We will use it also for the join of geometric entities in 3D, namely 3D points and 3D lines. (2) We will also use it for the corresponding algebraic entities. Thus we could have written in (5.14) the expression $\mathbf{l} = \mathbf{x} \wedge \mathbf{y}$, keeping in mind how the operation is performed algebraically. Applying the wedge to two 3-vectors therefore is identical to determining their cross product, independently of what the two vectors represent.

*overloading of $\wedge$ and $\cap$*

A similar reasoning leads to the homogeneous coordinates of the intersection point,

$$\chi = l \cap m : \quad \mathbf{x} = \mathbf{l} \times \mathbf{m} = \mathsf{S}(\mathbf{l})\mathbf{m}, \tag{5.16}$$

of two lines $l(\mathbf{l})$ and $m(\mathbf{m})$ given with homogeneous coordinates, where the sign $\cap$ (read: cap) indicates the intersection.

A second remark on notation: It would be more consistent to use the sign $\vee$ for the intersection. We found that in longer expressions it is difficult to distinguish visually between the $\wedge$-symbol and the $\vee$-symbol. Therefore, we intentionally use the sign $\cap$ for the intersection, which corresponds to the sign for the intersection of sets. This appears intuitive, as the intersection point is the set-intersection of the two lines, taken as the set

of infinitely many points. Again we will overload the symbol both for 3D entities, namely 3D lines and planes, and for algebraic entities.
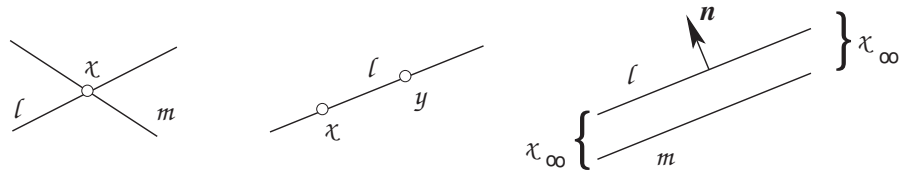

Fig. 5.4 Intersection $x = l \cap m$ **(left)** and join $l = x \wedge y$ **(centre)** of two geometric entities. The intersection of two parallel lines **(right)** leads to the point at infinity x_∞. The figure indicates this point to be the result of two different limiting processes. Given the direction $[u, v]^\mathsf{T}$ of the line, we may end up with $\mathbf{x}_\infty = \lim_{w \downarrow 0} [u, v, w]^\mathsf{T} = [u, v, 0]^\mathsf{T}$. But due to homogeneity, we also have $-\mathbf{x}_\infty = \lim_{w \uparrow 0} [u, v, w]^\mathsf{T} = \lim_{-w \downarrow 0} [-u, -v, -w]^\mathsf{T} = [-u, -v, 0]^\mathsf{T}$, the vector pointing in the opposite direction. In Sect. 9, p. 343, oriented projective geometry, we will distinguish between these two directions

If all vectors are spherically normalized, we arrive at a very intuitive interpretation of the construction equations, see Fig. 5.5. 2D points correspond to and are represented as points on the unit sphere, whereas 2D lines correspond to great circles on the unit sphere and are represented as unit normals, thus also as points on the unit sphere. The two constructions read as

$$\mathbf{l}^s = \mathsf{N}(\mathbf{x}^s \times \mathbf{y}^s) \qquad \text{and} \qquad \mathbf{x}^s = \mathsf{N}(\mathbf{l}^s \times \mathbf{m}^s), \tag{5.17}$$

which can be derived geometrically from the two graphs in Fig. 5.5.
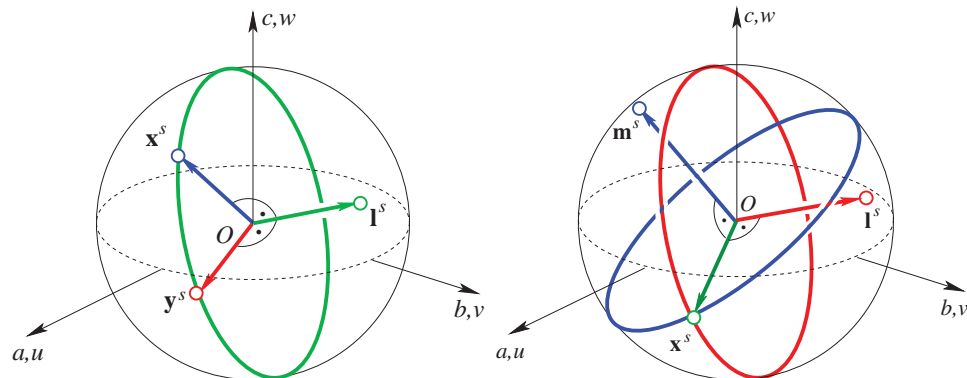
Fig. 5.5 Join of two points and intersection of two lines on the projective planes $\mathbb{P}^2$ and its dual plane $\mathbb{P}^{*2}$ superimposed on the same unit sphere (for the definition of $\mathbb{P}^{*2}$ cf. (5.38), p. 209). **Left:** The 2D line l joining any two 2D points $\mathbf{x}^s, \mathbf{y}^s$ on the projective plane is the great circle through these points. The normal $\mathbf{l}^s$ of the plane containing the great circle is determined by the normalized cross product of the two homogeneous coordinate vectors. **Right:** The 2D intersection point of any two 2D lines on the projective plane is the intersection of the two great circles defined by their normals $\mathbf{l}^s$ and $\mathbf{m}^s$. The direction of the intersection point $\mathbf{x}^s$ is the normalized cross product of the two normals of the planes containing the two great circles. If the intersection $\mathbf{x}^s$ lies on the equator, its last coordinate is zero, indicating the point is at infinity; thus, the two lines are parallel. Observe, the cross products are unique, a property which we will exploit when discussing oriented elements

Two parallel lines do not intersect in a point in the real plane but at infinity, which cannot be represented with inhomogeneous coordinates. However, the cross product of their homogeneous coordinates exists. This allows us to explicitly represent points at infinity.

Let the two lines, see Fig. 5.4, right, have the common normal $\boldsymbol{n}$ and two different distances d_1 and d_2 from the origin, with

$$\boldsymbol{n}^{\perp} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \boldsymbol{n} \tag{5.18}$$

perpendicular to the normal $\boldsymbol{n}$ of the lines. Then the homogeneous coordinates of their intersection point χ_{∞} are obtained from

$$\mathbf{x}_{\infty} = \begin{bmatrix} \boldsymbol{n} \\ -d_1 \end{bmatrix} \times \begin{bmatrix} \boldsymbol{n} \\ -d_2 \end{bmatrix} = \begin{bmatrix} (d_2 - d_1)\boldsymbol{n}^{\perp} \\ 0 \end{bmatrix} \cong \begin{bmatrix} \boldsymbol{n}^{\perp} \\ 0 \end{bmatrix}. \tag{5.19}$$

Thus, the first two components $[u, v]^{\mathsf{T}}$ of the 3-vector of a point at infinity,

point at infinity

$$\chi_{\infty}: \quad \mathbf{x}_{\infty} = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix}, \tag{5.20}$$

represent the direction towards the point at infinity, whereas the third component is zero.

Two points, $\chi_{\infty}([u_x, v_x, 0]^{\mathsf{T}})$ and $y_{\infty}([u_y, v_y, 0]^{\mathsf{T}})$, at infinity span what is called the *line at infinity*,

line at infinity

$$\ell_{\infty}: \quad \mathbf{l}_{\infty} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{5.21}$$

as the cross product yields $[0, 0, u_x v_y - u_y v_x]^{\mathsf{T}}$, which is proportional to $[0, 0, 1]^{\mathsf{T}}$. Any other point at infinity lies on the line at infinity.

All points with $\mathbf{x} \in \mathbb{R}^3 \setminus \mathbf{0}$, assuming proportional vectors represent the same point, are elements of the projective plane $\mathbb{P}^2$.[2] Reasoning with such projective elements is at the heart of projective geometry.

projective plane

All lines with $\mathbf{l} \in \mathbb{R}^3 \setminus \mathbf{0}$ are elements of the corresponding dual projective plane. This corresponds to the notion of a vector space for points and its dual for its linear forms. We will exploit the concept of duality between points and lines and generalize it to both 3D points and the corresponding transformations.

dual projective plane

Observe, the coordinates of the line $\ell = \chi \wedge y$ are not the same as the coordinates of the line $\ell = y \wedge \chi$, but are their negatives, as the cross product is anti-symmetric. This allows us to distinguish between lines with different directions if we follow certain sign conventions. For example, if we assume points to be represented with positive third component, we can distinguish between the sign of the lines $\chi \wedge y$ and $y \wedge \chi$, as their normals differ by $180°$. If we consistently consider the sign conventions, we arrive at the *oriented projective geometry*, which is the topic of Chap. 9.

oriented projective geometry

The 2D coordinate system can be described by its origin χ_0 and its axes ℓ_x and ℓ_y, with coordinates identical to unit 3-vectors $\boldsymbol{e}_i^{[3]}$,

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \boldsymbol{e}_3^{[3]}, \qquad \mathbf{l}_x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \boldsymbol{e}_2^{[3]}, \qquad \mathbf{l}_y = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{e}_1^{[3]}. \tag{5.22}$$

Note that the x-axis seen as a line ℓ_x has the Euclidean normal $[0, 1]^{\mathsf{T}}$ and passes through the origin, therefore $\mathbf{l}_x = \boldsymbol{e}_2$, not $\mathbf{l}_x = \boldsymbol{e}_1$. We will discuss the elements of coordinate systems in detail in Sect. 5.9.

5.1.2.5 Duality of Points and Lines

Each geometric element, operation, and relation has what is called a *dual*, indicated by $\overline{(\cdot)}$. The concept of duality results from the underlying three-dimensional vector space $\mathbb{R}^3$ for

---

[2] Mathematically, this is the quotient space $\mathbb{P}^2 = (\mathbb{R}^3 \setminus \mathbf{0})/(\mathbb{R} \setminus \mathbf{0})$, indicating that all vectors $\mathbf{x} \in \mathbb{R}^3 \setminus \mathbf{0}$ are taken as equivalent if they are multiplied with some $\lambda \in \mathbb{R} \setminus \mathbf{0}$.

representing points, with the dual vector space $\mathbb{R}^{*3}$, which contains all linear forms $\mathbf{l}^\mathsf{T}\mathbf{x}$, represented by the vector $\mathbf{l}$. As the two spaces are isomorphic, there is a natural mapping $D : \mathbb{R}^3 \mapsto \mathbb{R}^{*3}$, namely the identity mapping $\mathbf{x} \mapsto \mathbf{l}$.

Given the point $\mathbf{x} = [u, v, w]^\mathsf{T}$, the line $\mathbf{l}$ which is dual to this point has the same coordinates as the point

$$\mathbf{l} = \overline{\mathbf{x}} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{5.23}$$

and vice versa.

Therefore, a given 3-vector $[r, s, t]^\mathsf{T}$ can be interpreted as either a 2D point $\mathbf{x} = [r, s, t]^\mathsf{T}$ or a 2D line $\mathbf{l} = [r, s, t]^\mathsf{T}$; they are dual to each other.

The point $\mathcal{X}$ and the dual line ℓ are positioned on opposite sides of the origin O, with distances d_{xO} and d_{lO} to the origin multiplying to 1, thus $d_{xO} \, d_{lO} = 1$ (Table 7.3, p. 298). The line through $\mathcal{X}$ and perpendicular to the line ℓ passes through the origin, see Fig. 5.6. We will see that this property transfers to 3D.
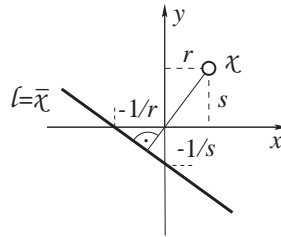
Fig. 5.6 Duality of points and lines in the plane. Point χ and line ℓ are dual w.r.t. each other. They have the same homogeneous coordinates $[r, s, t = 1]^\mathsf{T}$: $\chi(\boldsymbol{x} = [r, s]^\mathsf{T})$ and $\ell\,(rx + sy + 1 = 0)$, from which the intersection points $[-1/r, 0]$ and $[0, -1/s]$ with the axis can be derived

For spherically normalized homogeneous coordinates, we see from Fig. 5.3 that a point $\mathbf{x}^s$ and its dual line $\mathbf{l}^s = \overline{\mathbf{x}}^s$ are related by polarity on the sphere, cf. Sect. 5.1.2.3, p. 200. For more on duality cf. Sect. 5.6, p. 229.

5.1.2.6 Transformation of Points

Linear mappings of homogeneous coordinates can be used to represent classical transformations. For example, we have the translation $\mathcal{T}$ and the rotation $\mathcal{R}$,

$$\chi' = \mathcal{T}(\chi) : \quad \mathbf{x}' = \mathsf{T}\mathbf{x} \quad \text{with} \quad \mathsf{T}([t_x, t_y]^\mathsf{T}) = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \tag{5.24}$$

and

$$\chi' = \mathcal{R}(\chi) : \quad \mathbf{x}' = \mathsf{R}\mathbf{x} \quad \text{with} \quad \mathsf{R}(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5.25}$$

homogeneous matrix   which can easily be verified. Observe, the two 3×3 matrices are homogeneous entities: their multiplication with a scalar $\mu \neq 0$ does not change the transformation, as the resulting vector is multiplied with $\mu \neq 0$, leaving the resulting point unchanged.

Concatenation and inversion are obviously easy, since the geometric transformations *matrix* are represented as matrix vector products.
representation *for point*   Observe, the join of two points in (5.14) is also a matrix vector multiplication, suggesting that the skew matrix $\mathsf{S}(\mathbf{x})$ is a matrix representation of the point. We will generalize this

property for all basic geometric entities and transformations and derive a representation with homogeneous vectors and matrices.

We will see that a general linear mapping of homogeneous coordinates is straight line-preserving.

5.1.2.7 Variance Propagation and Estimation

All relations discussed so far are bilinear in the elements of the coordinates involved. Therefore, we may easily derive the Jacobians needed for variance propagation.

For example, from

$$\mathbf{l} = \mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x} = \mathsf{S}(\mathbf{x})\mathbf{y} = -\mathsf{S}(\mathbf{y})\mathbf{x} \tag{5.26}$$

we immediately obtain the two Jacobians

$$\frac{\partial(\mathbf{x} \times \mathbf{y})}{\partial \mathbf{y}} = \mathsf{S}(\mathbf{x}) \qquad \text{and} \qquad \frac{\partial(\mathbf{x} \times \mathbf{y})}{\partial \mathbf{x}} = -\mathsf{S}(\mathbf{y}). \tag{5.27}$$

The line coordinates are nonlinear functions of the point coordinates, namely sums of products. Following Sect. 2.7.6, p. 44, in a first approximation if the two points are stochastically independent with covariance matrices $\mathbf{\Sigma}_{xx}$ and $\mathbf{\Sigma}_{yy}$, we obtain the covariance matrix of the joining line,

$$\mathbf{\Sigma}_{ll} = \mathsf{S}(\boldsymbol{\mu}_x)\mathbf{\Sigma}_{yy}\mathsf{S}^{\mathsf{T}}(\boldsymbol{\mu}_x) + \mathsf{S}(\boldsymbol{\mu}_y)\mathbf{\Sigma}_{xx}\mathsf{S}^{\mathsf{T}}(\boldsymbol{\mu}_y). \tag{5.28}$$

Of course, we will need to discuss the meaning of the covariance matrix of a homogeneous entity and the degree of approximation resulting from the homogeneity of the representation.

Finally, we will discuss estimation techniques for homogeneous entities: we may use the homogeneity of the constraints to advantage to obtain approximate values. For example, let us assume N points $\boldsymbol{\chi}_n$, $n = 1, ..., N$, are given and we want to determine a best fitting straight line. Due to measurement deviations, the points and the unknown line will not satisfy the constraints $\mathbf{x}_n^{\mathsf{T}}\mathbf{l} = 0$ but will result in some residual $\mathbf{x}_n^{\mathsf{T}}\mathbf{l} = w_n$. Minimizing the length of the vector $\boldsymbol{w} = [w_1, ..., w_n, ..., w_N]^{\mathsf{T}}$, i.e., minimizing the sum of squared residuals $\boldsymbol{w}^{\mathsf{T}}\boldsymbol{w} = \sum_{n=1}^{N} w_n^2$ w.r.t. the line parameters under the constraint $|\mathbf{l}| = 1$, leads to minimizing the Rayleigh ratio

$$r = \frac{\mathbf{l}^{\mathsf{T}}\left(\sum_{n=1}^{N} \mathbf{x}_n\mathbf{x}_n^{\mathsf{T}}\right)\mathbf{l}}{\mathbf{l}^{\mathsf{T}}\mathbf{l}} \to \min, \tag{5.29}$$

which is known to be equivalent to solving an eigenvalue problem. As this method does not take the possibly different uncertainties of the points into account, we also need to discuss statistically optimal estimates. The special structure of the constraints will simplify the setup of the corresponding estimation problem.

5.2 Homogeneous Representations of Points and Lines in 2D

This section gives formal definitions of homogeneous coordinates of 2D points and 2D lines. It completes the descriptions of the concepts given so far for 2D space: the 3-vectors establishing the projective plane for points and the dual projective plane for lines. Both contain points and lines at infinity.